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A Note on a Differentiation Type Property of the Entropy of an Automorphism

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Let T be an automorphism of the nonatomic Lebesgue space (X, Σ, μ). Let $Z = Z(\Sigma)$ be the space of all Σ-measurable partitions of X possessing finite entropy in which two partitions are identified if they coincide modulo 0. For $\alpha, \beta \in Z$, Rohlin defined the distance $d(\alpha, \beta) = H(\alpha \mid \beta) + H(\beta \mid \alpha)$. Then (Z, d) is a complete metric space and $h(T, \cdot): Z \to [0, \infty)$ is a uniformly continuous function. Since Z is not naturally endowed with the structure of a linear space, one cannot properly define the notion of a derivative of h. The purpose of this note is to prove, using a recent result of Ornstein, a differentiation type property of h.

For definitions and properties of entropy see [4] or [5]. If ξ is a partition of X, we let ξ^T and ξ^{-T} denote the (possibly uncountable) partitions $\bigvee_{-\infty}^{\infty} T^i\xi$ and $\bigvee_{-\infty}^{-1} T^i\xi$, respectively.

DEFINITION. Given $\alpha, \beta \in Z$, let

$$Q(\alpha, \beta) = \frac{| h(T, \alpha) - h(T, \beta)|}{d(\alpha, \beta)}.$$

We will say that a partition $\alpha \in Z$ *possesses property* D if, given any $c \in [0, 1]$, there exists a sequence of partitions $\beta_i \in Z$ for which $Q(\alpha, \beta_i) \to c$ and $d(\alpha, \beta_i) \to 0$.

THEOREM. 1. *Let T be an ergodic automorphism having positive entropy and suppose $0 < h(T, \alpha) < h(T)$. Then α possesses property* D.

2. *If T is mixing and $h(T, \alpha) = 0$, then α possesses property* D.

The proof follows from a sequence of lemmas. The first, due to Ornstein (Lemma 5′ of [3]), was proven with the hypothesis that T be mixing. Using several minor modifications it can be shown that ergodicity of T is sufficient.

LEMMA 1. *Let T be an ergodic automorphism of X having positive entropy. Let β be a finite partition of X having jointly independent orbit $T^i\beta$, $-\infty < i < \infty$. Given $0 < k < h(T) - H(\beta)$, there exists a finite partition λ, also having jointly independent orbit $T^i\lambda$, such that λ^T and β^T are independent and $H(\lambda) = k$.*

LEMMA 2. *If $h(T, \alpha \vee \beta) = h(T, \alpha) + h(T, \beta)$ and $\lambda \leqslant \alpha^T$, $\omega \leqslant \beta^T$, where $\alpha, \beta, \lambda, \omega \in Z$, then $h(T, \lambda \vee \omega) = h(T, \lambda) + h(T, \omega)$.*

Proof. We may assume that $\omega \leqslant \beta$ (by replacing, if necessary, β by $\beta \vee \omega$). Given $\epsilon > 0$, there exists an integer N such that

$$H(\alpha^n) + H(\beta^n) - H(\alpha^n \vee \beta^n) < n\epsilon \qquad \text{whenever} \quad n > N.$$

Hence $H(\alpha^n) - H(\alpha^n \mid \omega^n) \leqslant H(\alpha^n) - H(\alpha^n \mid \beta^n) < n\epsilon$ for all $n > N$, and so $h(T, \alpha \vee \omega) = h(T, \alpha) + h(T, \omega)$. Repeat this argument once more to obtain the lemma.

LEMMA 3. *If $h(T, \alpha \vee \beta) = h(T, \alpha) + h(T, \beta)$, α and $\beta \in Z$, and n is a positive integer, then $h(T^n, \alpha \vee \beta) = h(T^n, \alpha) + h(T^n, \beta)$.*

Proof. If λ is any partition, let λ^n denote $\bigvee_0^n T^i\lambda$. Since $(\alpha^n \vee \beta^n)^{T^n} = (\alpha \vee \beta)^T$, we have

$$h(T^n, \alpha^n \vee \beta^n) = nh(T, \alpha \vee \beta) = n(h(T, \alpha) + h(T, \beta)) = h(T^n, \alpha^n) + h(T^n, \beta^n).$$

Since $\alpha \leqslant \alpha^n$, $\beta \leqslant \beta^n$, Lemma 2 implies that

$$h(T^n, \alpha \vee \beta) = h(T^n, \alpha) + h(T^n, \beta).$$

LEMMA 4. *If $h(T, \alpha \vee \beta) = h(T, \alpha) + h(T, \beta)$, α and $\beta \in Z$, and β generates a Bernoulli factor, then α^T and β^T are independent.*

Proof. Since $h(T, \beta) = h(T, \mu)$ whenever $\beta^T = \mu^T$, we can assume that β has jointly independent orbit, $T^i\beta$ $(-\infty < i < \infty)$. Applying Lemma 6.3 of [4], we have

$$\begin{aligned} h(T^n, \alpha^{n-1} \vee \beta^{n-1}) &= h(T^n, \alpha^{n-1}) + H(\beta^{n-1} | (\alpha^{n-1})^{T^n} \vee (\beta^{n-1})^{-T^n}) \\ &= h(T^n, \alpha^{n-1}) + H(\beta^{n-1} | \alpha^T \vee \beta^{-T}). \end{aligned}$$

Noting that $h(T, \alpha^{n-1} \vee \beta^{n-1}) = h(T, \alpha^{n-1}) + h(T, \beta^{n-1})$ and applying Lemma 3, we obtain $h(T^n, \alpha^{n-1} \vee \beta^{n-1}) = h(T^n, \alpha^{n-1}) + h(T^n, \beta^{n-1})$. Hence $h(T^n, \beta^{n-1}) = H(\beta^{n-1} | \alpha^T \vee \beta^{-T})$. Since the orbit of β^{n-1} under

T^n is jointly independent, $H(\beta^{n-1}) = H(\beta^{n-1} \mid \alpha^T \vee \beta^{-T})$, and so β^{n-1} is independent of α^T for any n and the proof of the lemma is complete.

LEMMA 5. *Let $\pi(T)$ denote the Pinsker kernel of T, and suppose $\beta \in Z$ generates a Bernoulli factor. Then $\pi(T)$ and β^T are independent.*

Proof. Suppose $\alpha \leqslant \pi(T)$ has finite entropy, $H(\alpha)$. From

$$h(T, \beta) \leqslant h(T, \alpha \vee \beta) \leqslant h(T, \alpha) + h(T, \beta) = h(T, \beta),$$

we have $h(T, \alpha \vee \beta) = h(T, \alpha) + h(T, \beta)$. Applying Lemma 4 we achieve the desired result.

LEMMA 6. *If (X, T) is a Bernoulli shift, then $N = \{\xi\colon h(T, \xi) = h(T)\}$ is nowhere dense.*

Proof. Since $h(T, \cdot)$ is continuous on Z, N is closed. Given $\xi \in N$ and $\epsilon > 0$, choose a finite partition β which generates the full σ-algebra and has jointly independent orbit $T^i\beta$, $-\infty < i < \infty$.

There exist a positive integer m and a partition ψ such that $d(\xi, \psi) < \epsilon/2$ and $\psi \leqslant \bigvee_{-m}^{m} T^i\beta$. By perturbing β slightly, one can find a sequence of partitions $\beta_i \to \beta$ such that each β_i has k atoms and $H(\beta_i) < H(\beta)$ for all i. Now $d(\bigvee_{-m}^{m} T^j\beta_i, \bigvee_{-m}^{m} T^j\beta) \leqslant (2m+1)\, d(\beta_i, \beta)$. As a consequence we can select a sequence of partitions $\psi_i \leqslant \bigvee_{-m}^{m} T^j\beta_i$ such that $\psi_i \to \psi$. In particular, there exists a K for which $d(\psi_K, \psi) < \epsilon/2$. Observe that

$$h(T, \psi_K) \leqslant h\left(T, \bigvee_{-m}^{m} T^j\beta_K\right) = h(T, \beta_K) \leqslant H(\beta_K) < H(\beta) = h(T).$$

Since $d(\psi_K, \xi) \leqslant d(\psi_K, \psi) + d(\psi, \xi) < \epsilon$, we obtain our result.

Proof of Theorem. 1. First assume $0 < c \leqslant 1$. Applying Sinai's theorem we can choose a partition $\beta \leqslant \alpha^T$ which generates a Bernoulli shift of entropy $h(T, \alpha)$. Lemma 1 allows us to select a sequence λ_j of finite partitions such that for each j the orbit $T^i\lambda_j$, $-\infty < i < \infty$, is jointly independent, $H(\lambda_j) < 1/j$, and $\lambda_j{}^T$ is independent of β^T. Hence $h(T, \lambda_j \vee \beta) = h(T, \lambda_j) + h(T, \beta)$. This implies that

$$\begin{aligned} h(T, \lambda_j \vee \alpha) &\leqslant h(T, \lambda_j) + h(T, \alpha) \\ &= h(T, \lambda_j) + h(T, \beta) = h(T, \lambda_j \vee \beta) \leqslant h(T, \lambda_j \vee \alpha). \end{aligned}$$

We have $h(T, \lambda_j \vee \alpha) = h(T, \lambda_j) + h(T, \alpha)$. Since λ_j generates a Bernoulli factor, Lemma 4 implies that λ_j^T is independent of α^T. Let Γ be the σ-subalgebra of Σ generated by atoms of the orbit of β. Since (X, Γ, μ) is also a Lebesgue space, we know that $Z(\Gamma)$ is path-connected. Consider the function $H(\cdot \mid \alpha)\colon Z(\Gamma) \to \mathbb{R}$. Since $H(\nu \mid \alpha) = 0$, where ν denotes the trivial partition, and $H(\beta^n \mid \alpha) \uparrow \infty$ as n increases, we can select a sequence of partitions $\psi_j \in Z(\Gamma)$ such that

$$H(\psi_j \mid \alpha) = \frac{1-c}{c} H(\lambda_j).$$

Let $K_j = \lambda_j \vee \psi_j \vee \alpha$. Then

$$\begin{aligned} d(K_j, \alpha) &= H(\lambda_j \vee \psi_j \mid \alpha) = H(\lambda_j \mid \psi_j \vee \alpha) + H(\psi_j \mid \alpha) = H(\lambda_j) + \frac{1-c}{c} H(\lambda_j) \\ &= \frac{1}{c} H(\lambda_j) \to 0 \qquad \text{as } j \to \infty. \end{aligned}$$

Observe that

$$\begin{aligned} \frac{|h(T, K_j) - h(T, \alpha)|}{d(K_j, \alpha)} &= \frac{h(T, \lambda_j \vee \alpha) - h(T, \alpha)}{1/c\, H(\lambda_j)} \\ &= c\, \frac{h(T, \lambda_j) + h(T, \alpha) - h(T, \alpha)}{H(\lambda_j)} = c\, \frac{h(T, \lambda_j)}{H(\lambda_j)} = c. \end{aligned}$$

In case $c = 0$, just choose a sequence of nontrivial partitions $\alpha_i \leqslant \alpha^T$ such that $H(\alpha_i) \downarrow 0$. Let $K_i = \alpha \vee \alpha_i$. Then $Q(K_i, \alpha) = 0$ and $d(K_i, \alpha) \to 0$.

2. If $\pi(T) \neq \nu$ and T is mixing, then $\pi(T)$ is nonatomic. Thus, at least in the case $c = 0$, one can repeat the same argument as in part 1.

Suppose $1 \geqslant c > 0$ and $\alpha \in \pi(T)$. If we replace $Z(\Gamma)$ by $Z(\pi(T))$, the argument of part 1 is applicable.

COROLLARY. *If (X, T) is a Bernoulli shift, then there does not exist an open set U of Z on which $h(T, \cdot)$ assumes a constant value.*

Proof. This is an immediate consequence of Lemma 6 and the theorem.

Note added in proof. Lemma 4 is actually a special case of a result due to Berg [1].

References

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